Solution to Assignment 10

Supplementary Problems

1. A vector field $\mathbf{F}$ is called radial if $\mathbf{F}(x,y,z) = f(r)(x,y,z),\ r = |(x,y,z)|$, for some function f. Show that every radial vector field is conservative. You may assume it is C^1 in $\mathbb{R}^3$.

   Solution. Let $\Phi(x,y,z)$ be the potential. Since f is radially symmetric, we believe that Φ is also radially symmetric. Let $\Phi(x,y,z) = \varphi(r),\ r = \sqrt{x^2+y^2+z^2}$. We have

   $$\frac{\partial \Phi}{\partial x} = \varphi'(r)\frac{x}{r}, \quad \frac{\partial \Phi}{\partial y} = \varphi'(r)\frac{y}{r}, \quad \frac{\partial \Phi}{\partial z} = \varphi'(r)\frac{z}{r} \ .$$

   By comparison, we see that Φ is a potential for $\mathbf{F}$ if $\varphi'(r)/r = f(r)$. Therefore,

   $$\varphi(r) = \int_0^r tf(t)\,dt \ ,$$

   is a potential for $\mathbf{F}$.

2. Let $F = (P,Q)$ be a C^1-vector field in $\mathbb{R}^2$ away from the origin. Suppose that $P_y = Q_x$. Show that for any simple closed curve C enclosing the origin and oriented in positive direction, one has

   $$\oint_C Pdx + Qdy = \lim_{\varepsilon\to 0} \varepsilon \int_0^{2\pi} \left[-P(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + Q(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta\right] d\theta \ .$$

   What happens when C does not enclose the origin?

   Solution. Let C_ε be the circle entered at the origin with radius ε which is so small to be enclosed by C. Connect C to C_ε by a path C' so that $C, \pm C'$, and C_ε form a closed path lying outside the origin and bounding a set on which the compatibility conditions are satisfied. By Green's theorem the line integral over C is equal to the line integral over C_ε, and that is it.

   The line integral vanishes when C does not include the origin.

3. We identity the complex plane with $\mathbb{R}^2$ by $x+iy \mapsto (x,y)$. A complex-valued function f has its real and imaginary parts respectively given by $u(x,y) = Ref(z)$ and $v(x,y) = Imf(z)$. Note that u and v are real-valued functions. The function f is called differentiable at z if

   $$\frac{df}{dz}(z) = \lim_{w\to 0} \frac{f(z+w)-f(z)}{w} \ ,$$

   exists.

   (a) Show that f is differentiable at z implies that the partial derivatives of u and v exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, hold. Hint: Take $w = h, ih$, where $h \in \mathbb{R}$ and then let $h \to 0$.

   Solution. Identify z with (x,y). As f is differentiable at z, for real h,

   $$\begin{aligned} f'(z) &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\left(\frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}\right) \\ &= \lim_{h\to 0}\frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0}\frac{v(x+h,y)-v(x,y)}{h} \ . \end{aligned}$$

Using the fact that $a_n + ib_n \to a + ib$ if and only if $a_n \to a$ and $b_n \to b$ (here $f'(z) = a + ib$), we see that $\partial u/\partial x$ and $\partial v/\partial x$ exists and

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}(x, y) = f'(z) \ .$$

Next, we consider purely imaginary ih,

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{ih} = \lim_{h\to 0}\left(-i\frac{u(x, y+h) - u(x, y)}{h} + \frac{v(x, y+h) - v(x, y)}{h}\right) \\ &= -i\lim_{h\to 0}\frac{u(x, y+h) - u(x, y)}{h} + \lim_{h\to 0}\frac{v(x, y+h) - v(x, y)}{h} \ . \end{aligned}$$

As before, $\partial u/\partial y$ and $\partial v/\partial y$ exists and

$$-i\frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y) = f'(z) \ .$$

By comparison, we have $\partial v/\partial y = \partial u/\partial x$ and $-\partial u/\partial y = \partial v/\partial x$ at (x, y).

(b) Propose a definition of $\int_C f\,dz$, where C is an oriented curve in the plane, in terms of the line integrals involving u and v.

Solution. Formally we have $f dz = (u + iv)(dx + idy) = udx - vdy + i(vdx + udy)$. So, we define

$$\int_C f\,dz = \int_C udx - vdy + i\int_C vdx + udy \ .$$

Note that the right hand side are two line integrals.

(c) Suppose that f is differentiable everywhere in $\mathbb{C}$. Show that for every simple closed curve C,

$$\oint_C f\,dz = 0 \ .$$

Solution. Use (a) we see that $P = u, Q = -v$ as well as $P = v, Q = u$ satisfy the compatibility conditions. Hence, by Green's theorem,

$$\oint_C f\,dz = 0 \ .$$

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The conclusion in (c) is called Cauchy's theorem. It is a fundamental result in complex analysis.

Exercises 16.4

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve C.

14. $\mathbf{F} = \left(\tan^{-1}\frac{y}{x}\right)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j}$

C: The boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2$, $0 \le \theta \le \pi$

Using Green's Theorem

Calculating Area with Green's Theorem If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

The reason is that by Equation (4), run backward,

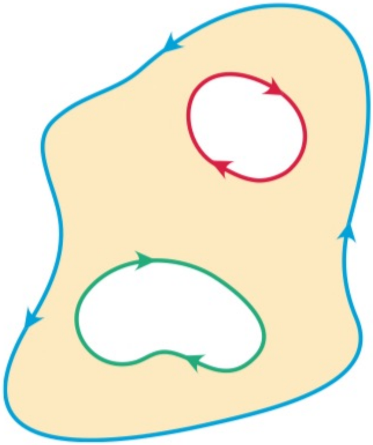
$$\text{Area of } R = \iint_R dy\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx$$

$$= \oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx.$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

27. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 \le t \le 2\pi$

39. Regions with many holes Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (see accompanying figure).



a. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n}\, ds.$$

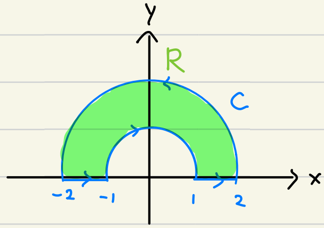
b. Let K be an arbitrary smooth, simple closed curve in the plane that does not pass through $(0, 0)$. Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n}\, ds$$

has two possible values, depending on whether $(0, 0)$ lies inside K or outside K.

Q14 $M(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$; $\frac{\partial M}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}$; $\frac{\partial M}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$

$N(x,y)=\ln(x^2+y^2)$; $\frac{\partial N}{\partial x}=\frac{1}{x^2+y^2}\cdot 2x=\frac{2x}{x^2+y^2}$; $\frac{\partial N}{\partial y}=\frac{1}{x^2+y^2}\cdot 2y=\frac{2y}{x^2+y^2}$



$\therefore \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=\frac{x}{x^2+y^2}$; $\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}=\frac{y}{x^2+y^2}$

$$\therefore \text{Flux}=\iint_R \frac{y}{x^2+y^2}\,dA=\int_0^{\pi}\int_1^2 \frac{r\sin\theta}{r^2}(r\,dr\,d\theta)$$

$$=\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_1^2 dr\right)=[-\cos\theta]_0^{\pi}[r]_1^2=2$$

$$\text{Circulation}=\iint_R \frac{x}{x^2+y^2}\,dA=\int_0^{\pi}\int_1^2 \frac{r\cos\theta}{r^2}(r\,dr\,d\theta)$$

$$=\left(\int_0^{\pi}\cos\theta\,d\theta\right)\left(\int_1^2 dr\right)=[\sin\theta]_0^{\pi}[r]_1^2=0$$

Q27 $x=x(t)=\cos^3 t$; $dx=-3\cos^2 t\sin t\,dt$

$y=y(t)=\sin^3 t$; $dy=3\sin^2 t\cos t\,dt$

$$\therefore \text{Area}=\frac{1}{2}\int_0^{2\pi}(\cos^3 t)(3\sin^2 t\cos t\,dt)-(\sin^3 t)(-3\cos^2 t\sin t\,dt)$$

$$=\frac{1}{2}\int_0^{2\pi}3\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)dt$$

$$=\frac{3}{8}\int_0^{2\pi}\sin^2 2t\,dt=\frac{3}{16}\int_0^{2\pi}(1-\cos 2t)dt$$

$$=\frac{3}{16}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}=\frac{3\pi}{8}$$

Q39 (a) Let $\nabla f(x,y) = \frac{2x}{x^2+y^2}\vec{i} + \frac{2y}{x^2+y^2}\vec{j} = M(x,y)\vec{i} + N(x,y)\vec{j}$.

Case 1: C has anti-clockwise orientation. Then C can be parametrized as

$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$, where $0 \le t < 2\pi$.

Then $M(\vec{r}(t)) = \frac{2a\cos t}{a^2(\cos^2 t + \sin^2 t)} = \frac{2}{a}\cos t$; $N(\vec{r}(t)) = \frac{2a\sin t}{a^2(\cos^2 t+\sin^2 t)} = \frac{2}{a}\sin t$.

$\vec{r}'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$; $\vec{n}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$\therefore \oint_C \nabla f \cdot \vec{n}\, ds = \int_0^{2\pi} \left(\left(\frac{2}{a}\cos t\right)\cdot(a\cos t) + \left(\frac{2}{a}\sin t\right)\cdot(a\sin t) \right) dt = 2\int_0^{2\pi} dt = 4\pi$ //

Case 2: C has clockwise orientation. Let C' be the circle with anticlockwise orientation.

Then $\oint_C \nabla f \cdot \vec{n}\, ds = -\oint_{C'} \nabla f \cdot \vec{n}\, ds = -4\pi$ //

Rmk The numerical answer 0, appeared in the textbook "Thomas' Calculus" (13 Edition) is incorrect.

(b) Case 1: $(0,0)$ does not lie inside K. Let D be the region bounded by K.

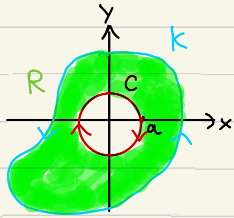
Then by Green's Theorem, $\oint_K \nabla f \cdot \vec{n}\, ds = \pm \iint_D (M_x + N_y)\, dA$ (where $(\pm)$ sign depends on the orientation of K)

$$= \pm \iint_D \left(\frac{(x^2+y^2)\cdot 2 - 2x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)\cdot 2 - 2y(2y)}{(x^2+y^2)^2} \right) dA = 0 \,/\!/$$

Case 2: $(0,0)$ lies inside K. Choose $a > 0$ sufficiently small such that C lies inside K.

K has anticlockwise orientation and C has clockwise orientation.

Let R be the region lying between K and C.



Then since R does not enclose $(0,0)$,

$$0 = \iint_R (M_x + N_y)\, dA \quad \text{(by case 1)}$$

$$= \oint_K \nabla f \cdot \vec{n}\, ds + \oint_C \nabla f \cdot \vec{n}\, ds \quad \text{(by Green's Theorem)}$$

$$= \oint_K \nabla f \cdot \vec{n}\, ds + (-4\pi) \quad \text{(by (a), case 2)}. \quad \therefore \oint_K \nabla f \cdot \vec{n}\, ds = 4\pi.$$

Combining both cases, $$\oint_K \nabla f \cdot \vec{n}\, ds = \begin{cases} 0, & \text{if } (0,0) \text{ does not lie inside } K. \\ 4\pi, & \text{if } (0,0) \text{ lies inside } K. \end{cases}$$